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THE AXISYMMETRIC STATIC PROBLEM OF THERMOELASTICITY FOR A MULTILAYERED CYLINDER†

YU. M. KOLYANO (dec.), B. V. PROTSYUK and V. M. SINYUTA

L'vov

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A method of solving the axisymmetric static problem of thermoelasticity based on the use of generalized functions is proposed for a multilayered unbounded solid cylinder free of external loads, through whose surface convective heat exchange occurs with a variable heat transfer coefficient.

1. EQUATIONS WITH DISCONTINUOUS AND SINGULAR COEFFICIENTS OF THE TWO-DIMENSIONAL STATIC PROBLEM OF THERMOELASTICITY OF MULTILAYER CYLINDERS

CONSIDER a cylinder of circular transverse cross-section, free from external loads, composed of an arbitrary number of concentrically distributed layers with different physical and mechanical characteristics. The cylinder is heated by convective heat transfer from the surrounding medium of variable temperature. We will assume that the cylinders are in ideal thermomechanical contact with each other, and that the heat transfer coefficient is a function of the axial coordinate.

We will write the physical and mechanical characteristics of a multilayered cylinder as a single whole in the form [1]

$$p(r) = p_1 + \sum (p_{k+1} - p_k) S(r - r_k), \quad S(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leqslant 0 \end{cases} \tag{1.1}$$

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Here $S(x)$ is the Heaviside function [2], r_k, p_k are the outer radius and the characteristic of the kth layer, respectively and n is the number of layers. Here and henceforth, unless otherwise stated, summation is carried out over k from $k=1$ to $k=n-1$.

The heat conduction equation of the inhomogeneous body in question has the form [1]

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\lambda^t(r)\frac{\partial t}{\partial r}\right]+\lambda^t(r)\frac{\partial^2 t}{\partial z^2}=0 \tag{1.2}$$

where the thermal conductivity $\lambda^t(r)$ is given by formula (1.1).

Using reasoning analogous to that in [1] and taking into account the relation between the generalized and classical derivative and the conditions of ideal thermal contact

$$t\,|_{r=r_k+0}=t\,|_{r=r_k-0},\quad \partial t/\partial r\,|_{r=r_k+0}=K_k^\lambda \partial t/\partial r\,|_{r=r_k-0} \tag{1.3}$$

we obtain the following relation from (1.2) [$\delta(x)$ is the delta function]:

$$\Delta t-\sum(\partial t/\partial r\,|_{r=r_k+0}-\partial t/\partial r\,|_{r=r_k-0})\,\delta(r-r_k)=0 \tag{1.4}$$

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},\quad K_k^\lambda=\frac{\lambda_k^t}{\lambda_{k+1}^t}$$

Taking into account the second condition of (1.3), we can transform the equation to the form

$$\Delta t+\sum(1-K_k^\lambda)\,\partial t/\partial r\,|_{r=r_k-0}\,\delta(r-r_k)=0 \tag{1.5}$$

Equation (1.5) is equivalent to the equation of heat conduction for each layer and conditions of ideal thermal contact. Indeed,

$$t(r,z)=t_1(r,z)+\Sigma\,[t_{k+1}(r,z)-t_k(r,z)]\,S(r-r_k)$$

where $t_k(r,z)$ is the temperature of the kth layer of the cylinder.

Defining the generalized derivative functions $t(r,z)$ as in [2] and substituting them into Eq. (1.5), we obtain

$$\Delta t_1+\Sigma\,[\Delta t_{k+1}-\Delta t_k]\,S(r-r_k)+\Sigma\,\{[\partial t/\partial r\,|_{r=r_k+0}-$$
$$-K_k^\lambda\partial t/\partial r\,|_{r=r_k-0}]\,\delta(r-r_k)+[t_{k+1}\,|_{r=r_k+0}-t_k\,|_{r=r_k-0}]\,[r_k^{-1}\delta(r-r_k)+$$
$$+\delta'(r-r_k)]\}=0$$

From this it follows that [3]

$$\Delta t_k=0\quad\text{for}\quad r_{r-1}<r<r_k$$
$$t_{k+1}\,|_{r=r_k+0}=t_k\,|_{r=r_k-0},\quad \partial t\,|_{k+1}/\partial t\,|_{r=r_k+0}=K_k^\lambda\partial t_k/\partial r\,|_{r=r_k-0}$$

Analogous arguments and substitution of the known Duhamel–Neumann relations [4] into the equations of equilibrium yield the following system of equations:

$$\Delta u_r-\frac{u_r}{r^2}-\frac{\lambda+\mu}{\lambda+2\mu}\left(\frac{\partial^2 u_r}{\partial z^2}-\frac{\partial^2 u_z}{\partial z\partial r}\right)-\frac{\beta}{\lambda+2\mu}\frac{\partial t}{\partial r}+$$
$$+\Sigma\frac{1}{\lambda_{k+1}+2\mu_{k+1}}\left\{2(\mu_{k+1}-\mu_k)\frac{\partial u_r}{\partial r}+(\lambda_{k+1}-\lambda_k)e-\right. \tag{1.6}$$
$$\left.-(\beta_{k+1}-\beta_k)\,t\right\}\Big|_{r=r_k-0}\delta(r-r_k)=0$$

$$\Delta u_z + \frac{\lambda+\mu}{\mu}\frac{\partial e}{\partial z} - \beta\frac{\partial t}{\partial z} + \Sigma\gamma_k^{(1)}\left\{\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right\}\delta(r-r_k) = 0$$

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}, \quad \beta = \alpha^t(3\lambda+2\mu), \quad \gamma_k^{(1)} = \frac{\mu_{k+1}-\mu_k}{\mu_{k+1}}$$

Here u_r, u_z are the radial and axial displacements, respectively, λ, μ, α^t are the Lamé coefficients and temperature coefficient of linear expansion of the form (1.1).

We can also show that system (1.6) is equivalent to the system of equations of equilibrium in terms of displacements for every layer and conditions of ideal thermomechanical contact.

Note that Eq. (1.5) and system (1.6) can also be obtained using the associative, non-communicative product†

$$\begin{aligned} f(x)\,\delta(x-a) &= f(a+0)\,\delta(x-a) \\ \delta(x-a)\,f(x) &= f(a-0)\,\delta(x-a) \end{aligned} \tag{1.7}$$

and the rules of generalized differentiation of a product

$$[f(x)\,g(x)]' = f(x)\,g'(x) + f'(x)\,g(x) \tag{1.8}$$

where $f(x)$, $g(x)$ are functions that are sufficiently smooth over the whole region in question except a finite number of points, all of them with first-order discontinuities.

2. THE THERMAL STRESS STATE OF A MULTILAYER CYLINDER

In order to determine the temperature field and displacements caused by it, we will use Eq. (1.5) and system (1.6) with the following boundary conditions:

$$\begin{gathered} \lambda_n{}^t\,\partial t/\partial r + \alpha(z)(t - t_c(z)) = 0 \text{ when } r = r_n \\ t \neq \infty \text{ when } r = 0;\ t,\ \partial t/\partial z \to 0 \text{ when } z \to \pm\infty \end{gathered} \tag{2.1}$$

$$\sigma_{rr} = \sigma_{rz} = 0 \text{ when } r = r_n;\ u_r \neq \infty,\ u_z \neq \infty \text{ when } r = 0 \tag{2.2}$$

Here $t_c(z)$ is the temperature of the surrounding medium, $\alpha(z)$ is the heat transfer coefficient, and σ_{rr} and σ_{rz} are the normal and shear stresses.

Representing the heat transfer coefficient in the form $\alpha(z) = \alpha_1 + \alpha_0(z)$ and using Green's function [5], we obtain the solution of problem (1.5), (2.1)

$$t(r,z) = \int_0^\infty M_1(z,\eta)\,T(r,\eta)\,d\eta \tag{2.3}$$

$$M_1(z,\eta) = \frac{1}{\pi A}\int_{-\infty}^{\infty}[\alpha(\zeta)\,t_c(\zeta) - \alpha_0(\zeta)\,t(r_n,\zeta)]\cos\eta(z-\zeta)\,d\zeta$$

$$T(r,\eta) = I_0(\eta r) - \eta\Sigma(1-K_k{}^\lambda)\,r_k\psi_{0,0}(r,r_k)\,H_{1,k}^t S(r-r_k)$$

$$A = \lambda_n{}^t\eta H_{1,n}^t + \alpha_1 H_{0,n}^t$$

$$H_{i,k}^t = I_i(\eta r_k) - \eta\sum_{m=1}^{k-1}(1-K_m{}^\lambda)\,r_m\psi_{i,0}(r_k,r_m)\,H_{1,m}^t$$

$$\psi_{i,j}(x,y) = I_i(\eta x)\,K_j(\eta y) + (-1)^{i+j+1}K_i(\eta x)\,I_j(\eta y)$$

†Protsyuk B. V., Temperature fields and stresses in cylindrical multilayer bodies. Candidate dissertation, L'vov, 1983.

Here $I_j(x)$, $K_j(x)$ are modified Bessel functions of order j and $t(r_n, z)$ is the solution of the integral equation

$$t(r_n, z) = \int_0^\infty M_1(z, \eta) H^t_{0,n}\, d\eta$$

The constant α_1 lies within the range of variation of $\alpha(z)$. To solve problem (1.6), (2.2), we will express u_r, u_z in terms of the thermoelastic displacement potential

$$u_r = u + \partial\Phi/\partial r,\quad u_z = v + \partial\Phi/\partial z \tag{2.4}$$

and seek the functions Φ, u, v in the form

$$\Phi = \int_0^\infty M_1(z, \eta)\, \varphi(r, \eta)\, d\eta$$

$$u = \int_0^\infty M_1(z, \eta)\, U(r, \eta)\, d\eta, \qquad v = \int_0^\infty M_2(z, \eta)\, V(r, \eta)\, d\eta \tag{2.5}$$

$$M_2(z, \eta) = \frac{1}{\pi A} \int_{-\infty}^{\infty} [\alpha(\zeta)\, t_c(\zeta) - \alpha_0(\zeta)\, t(r_n, \zeta)] \sin \eta (z - \zeta)\, d\zeta$$

After substituting expressions (2.3)–(2.5) into (1.6), (2.2) and multiplying the first equation of (1.6) on the left by $(\lambda + 2\mu)/\mu$ we obtain, in accordance with (1.7), the equation for determining φ:

$$L_0\varphi = bT,\quad b = \beta/(\lambda + 2\mu) \tag{2.6}$$

and, respectively, a system of equations and boundary conditions for determining U, V:

$$L_1 U + \frac{\lambda + \mu}{\mu} \frac{d\varepsilon}{dr} + F_1 = 0, \quad L_0 V - \frac{\lambda + \mu}{\mu} \eta\varepsilon + F_2 = 0 \tag{2.7}$$

$$\frac{dU}{dr} + \frac{\lambda_n}{2\mu_n} \varepsilon = \frac{1}{r} \frac{d\varphi}{dr} - \eta^2\varphi \tag{2.8}$$

$$\frac{dV}{dr} - \eta U = 2\eta \frac{d\varphi}{dr} \text{ when } r = r_n; \quad U \neq \infty,\ V \neq \infty \text{ when } r = 0$$

where

$$L_0 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \eta^2, \quad L_1 = L_0 - \frac{1}{r^2}, \quad \varepsilon = \frac{dU}{dr} + \frac{U}{r} + \eta V$$

$$F_1(r, \eta) = -\frac{1}{\eta} \sum \sum_{m=1}^{2} \gamma_k^{(m)} H_k^{(m)} \delta(r - r_k), \quad \gamma_k^{(2)} = \frac{\lambda_{k+1} - \lambda_k}{\mu_{k+1}}$$

$$F_2(r, \eta) = \Sigma \gamma_k^{(1)} H_k^{(3)} \delta(r - r_k)$$

$$H_k^{(1)} = -2\eta \left(\frac{dU}{dr} - \frac{1}{r}\frac{d\varphi}{dr} + \eta^2\varphi \right)\Big|_{r=r_k-0} \tag{2.9}$$

$$H_k^{(2)} = -\eta\varepsilon\,|_{r=r_k-0}, \quad H_k^{(3)} = \left(\frac{dV}{dr} - \eta U - 2\eta \frac{d\varphi}{dr} \right)\Big|_{r=r_k-0}$$

Solving Eq. (2.6) we obtain

$$\varphi(r,\eta)=\frac{br}{2\eta}I_1(\eta r)-{}^1/_2\sum_{m=1}^{n-1}\Big\{(b_{m+1}-b_m)r_m{}^2f_1(r,r_m)+$$

$$+(1-K_m{}^\lambda)r_mH^t_{1,m}\Big[b_{m+1}f_2(r,r_m,r_m)+\sum_{l=m+1}^{n-1}(b_{l+1}-b_l)f_2(r,r_m,r_l)\Big]\Big\}$$

$$f_1(r,r_m)=[(I_0(\eta r_m)\psi_{0,0}(r,r_m)+I_1(\eta r_m)\psi_{0,1}(r,r_m)]S(r-r_m)$$

$$f_2(r,r_m,r_l)=[r\psi_{1,0}(r,r_m)-\eta r_l^2(\psi_{0,0}(r_l,r_m)\psi_{0,0}(r,r_l)+$$

$$+\psi_{1,0}(r_l,r_m)\psi_{0,1}(r,r_l))]S(r-r_l)$$

After some reduction using the rule (1.8) and the product (1.7), we reduce the system of equations (2.7) to the following system:

$$U=\frac{\lambda+2\mu}{\lambda+\mu}\frac{1}{\eta^3}\Big[\frac{d(L_0V)}{dr}+\frac{dF_2}{dr}+F_3\Big]+\frac{F_1}{\eta^2}+\frac{1}{\eta}\frac{dV}{dr} \tag{2.10}$$

$$L_0{}^2V=-\Big[\frac{\lambda+\mu}{\lambda+2\mu}\eta\Big(\frac{dF_1}{dr}+\frac{F_1}{r}+\eta F_2\Big)+L_0F_2+\frac{dF_3}{dr}+\frac{F_3}{r}-\frac{\mu}{\lambda+2\mu}\eta F_4\Big]$$

Here

$$F_3(r,\eta)=\Sigma\gamma_k^{(3)}H_k^{(2)}\delta(r-r_k),\quad F_4(r,\eta)=\eta\Sigma\gamma_k^{(3)}\gamma_k^{(4)}H_k^{(4)}\delta(r-r_k)$$

$$H_k^{(4)}=\Big(\frac{dV}{dr}+\eta U\Big)\Big|_{r=r_k-0},\quad \gamma_k^{(3)}=\frac{\lambda_{k+1}}{\mu_{k+1}}-\frac{\lambda_k}{\mu_k},\quad \gamma_k^{(4)}=\frac{\mu_k}{\lambda_k+2\mu_k} \tag{2.11}$$

The solution of the second equation of system (2.10) bounded at $r=0$, has the form (c_1, c_2 are unknown constants)

$$V={}^1/_2\Sigma[(r\psi_{1,1}(r,r_k)P_k^{(1)}-r_k\psi_{0,0}(r,r_k)P_k^{(2)})S(r-r_k)-$$

$$-\eta f_2(r,r_k,r_k)P_k^{(3)}]+c_1I_0(\eta r)+c_2rI_1(\eta r) \tag{2.12}$$

$$P_k^{(i)}=r_k\sum_{m=0}^{1}\omega_k^{(i+m)}H_k^{(i+m)}\ (i=1,3),\quad P_k^{(2)}=P_k^{(1)}+2\gamma_k^{(1)}H_k^{(3)}$$

$$\omega_k^{(1)}=\omega_k^{(3)}=(1-\gamma_{k+1}^{(4)})\gamma_k^{(1)},\quad \omega_k^{(2)}=(1-\gamma_{k+1}^{(4)})\gamma_k^{(2)}-\gamma_k^{(3)},\quad \omega_k^{(4)}=\gamma_{k+1}^{(4)}-\gamma_k^{(4)}$$

Using the representation

$$P_k^{(i)}=\sum_{j=1}^{3}c_jP_{k,j}^{(i)},\quad H_k^{(i)}=\sum_{j=1}^{3}c_jH_{k,j}^{(i)},\quad c_3=1 \tag{2.13}$$

from (2.12) and the first equation of system (2.10) we find the required relations for U and V. The quantities $H_{k,j}^{(i)}$ are found from the recurrence relations which are obtained by substituting the representations (2.13) and relations for U, V, into (2.9) and (2.11). The constants c_1, c_2 obtained from the boundary conditions (2.8), have the form

$$c_1=(g_{12}g_{23}-g_{13}g_{22})/D,\quad c_2=(g_{13}g_{21}-g_{23}g_{11})/D$$

$$D=g_{11}g_{22}-g_{21}g_{12},\quad g_{1j}=\frac{1}{2}(H_{n,j}^{(1)}+(\lambda_n/\mu_n)H_{n,j}^{(2)}),\quad g_{2j}=H_{n,j}^{(3)}$$

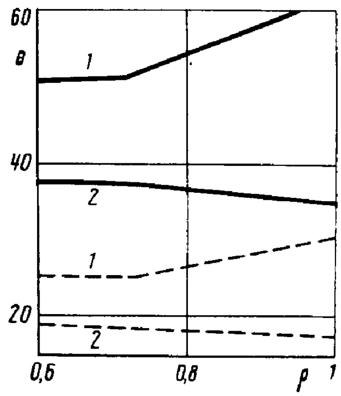

Fig. 1.

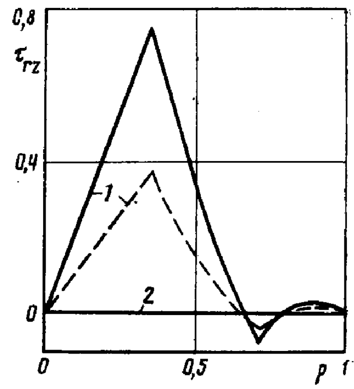

Fig. 2.

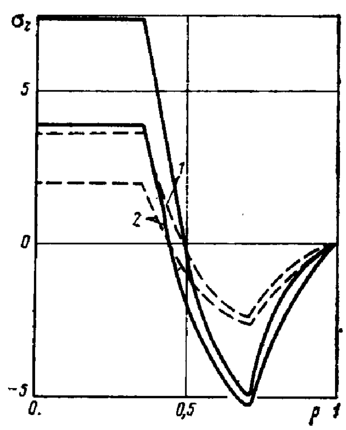

Fig. 3.

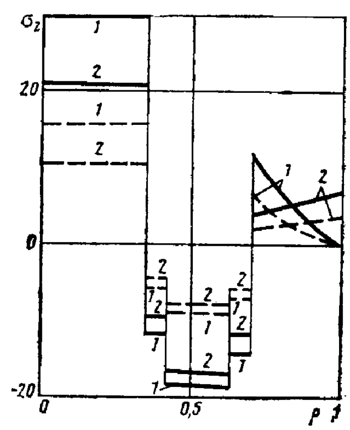

Fig. 4.

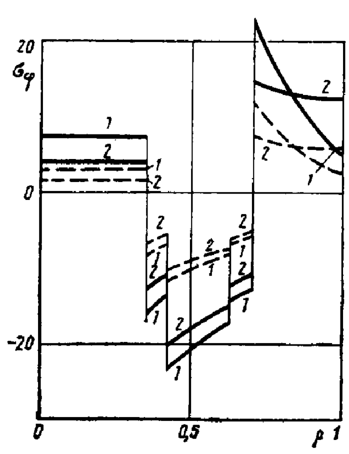

Fig. 5.

As an example for the case when the temperature of the surrounding medium and heat transfer coefficient vary according to the law

$$t_c(z) = t_0 N(z), \quad \alpha(z) = \alpha_1 + (\alpha_2 - \alpha_1) N(z)$$
$$N(z) = {}^1/_2 \operatorname{erfc}(20(|z/r_n| - 2))$$

we calculated the temperature field and corresponding temperature stresses in a five-layer cylindrical system for the following values of the thermoelastic and geometrical characteristics:

$$E_1 = 11\times10^{10}\ \mathrm{N/m^2},\ \nu_1 = 0.26,\ \alpha_1{}' = 0.25\times10^{-5}\ \mathrm{1/K},\ \lambda_1{}' = 80\ \mathrm{W/m\,K}$$
$$E_2 = 2.7\times10^{10}\ \mathrm{N/m^2},\ \nu_2 = 0.33,\ \nu_2{}' = 2.6\times10^{-5}\ \mathrm{1/K},\ \lambda_2{}' = 46.1\ \mathrm{W/m\,K}$$
$$E_3 = 11.1\times10^{10}\ \mathrm{N/m^2},\ \nu_3 = 0.35,\ \alpha_3{}' = 1.7\times10^{-5}\ \mathrm{1/K},\ \lambda_3{}' = 393.6\ \mathrm{W/m\,K}$$
$$E_4 = E_2,\ \nu_4 = \nu_2,\ \alpha_4{}' = \alpha_2{}',\ \lambda_4{}' = \lambda_2{}'$$
$$E_5 = 20.6\times10^{10}\ \mathrm{N/m^2},\ \nu_5 = 0.26,\ \alpha_5{}' = 1.1\times10^{-5}\ \mathrm{1/K},\ \lambda_5{}' = 6.3\ \mathrm{W/m\,K}$$
$$r_1 = 5\times10^{-3}\ \mathrm{m},\ r_2 = 6\times10^{-3}\ \mathrm{m},\ r_3 = 9\times10^{-3}\ \mathrm{m},\ r_4 = 10^{-2}\ \mathrm{m},\ r_5 = 1.4\times10^{-2}\ \mathrm{m}$$

Here E_k is Young's modulus and ν_k is Poisson's ratio of the kth layer.

The solid lines in Figs 1–5 show the results of computations for a variable heat transfer coefficient ($\alpha_1 = 100$ W/m^2 K, $\alpha_2 = 350$ W/m^2 K), and the dashed lines the case of a constant coefficient ($\alpha_1 = \alpha_2 = 100$ W/m^2 K). Figure 1 shows the dependence of dimensionless temperature $\theta = 10^2\, t/t_0$ and Figs 2–5 show the dependence of dimensionless shear and normal stresses

$$\tau_{rz} = 10^2 \frac{\sigma_{rz}}{\sigma_0}, \quad \sigma_r = 10^2 \frac{\sigma_{rr}}{\sigma_0}, \quad \sigma_z = 10^2 \frac{\sigma_{zz}}{\sigma_0}, \quad \sigma_\varphi = 10^2 \frac{\sigma_{\varphi\varphi}}{\sigma_0} \quad (\sigma_0 = \alpha_5{}^t E_5 t_0)$$

on $\rho = r/r_5$ for the following values of $z/r_5 = 0$ (curves 1) and $z/z_5 = 3$ (curves 2).

From the above graphs it follows that the values of the temperature and the absolute values of the stresses in a cylinder are approximately twice as large in the case of a variable heat transfer coefficient, as in the case of a constant coefficient. When the dimensionless axial coordinate z/r_5 increases from 0 to 3, the absolute values of temperature and stresses decrease everywhere except in the interval $0.85<\rho<1$ in which the axial stresses σ_z (Fig. 4) and annular (circumferential) stresses σ_φ (Fig. 5) increase; the largest stresses are the axial stresses σ_z in the first layer, while the largest compressive stresses are the annular (circumferential) stresses at the boundary between the third and fourth layers.

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